

Perturbations in stochastic inflation

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Abstract

The perturbative approach to stochastic inflation is used to determine the spectrum of density fluctuations and gravitational waves due to the coarse grained field. The amplitude of the curvature fluctuation spectrum, the spectral index and the running of the spectral index are in general found to be smaller than in the standard approach to inflation. Furthermore, the amount of non-gaussianity due to the second order perturbation is estimated.

1 Introduction

Inflation provides a mechanism in order to seed large scale structure formation. Quantum fluctuations of the inflaton are stretched beyond the horizon and are converted into classical perturbations. In the standard approach to calculating the spectrum of the resulting fluctuations expectation values of the quantum fluctuations are identified with statistical averages of classical perturbations. Thus the amplitude of fluctuations is determined by the vacuum expectation value of the quantum field. The details of the quantum to classical transition are neglected. It is assumed that perturbations are classical instantaneously at horizon exit. The assumption of a classical behaviour outside the horizon has been found to be justified, as it is related to the particular behaviour of the mode functions in a de Sitter space-time (see for example [1]).

In the stochastic approach to inflation the dynamics of the quantum to classical transition is effectively described by a classical noise. The scalar field is coarse grained. Thus the field modes are split into a subhorizon part and a superhorizon part. The superhorizon part constitutes the classical, homogeneous coarse grained field driven by the stochastic noise due to the subhorizon modes leaving the horizon. As the inflationary universe keeps expanding rapidly more and more short wavelength modes are stretched beyond the horizon and subsequently contribute to the coarse grained, classical field. The coarse graining scale is at least of the order of the horizon. As shown in [2] in the case of a scalar field in a de Sitter space-time, the effective dynamics of the classical, coarse grained field is described by a Langevin equation with a white noise term. However, it

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has been shown that the appearance of the white noise term is a direct consequence of the choice of window function to separate the sub- and superhorizon modes [3]. Furthermore, models of stochastic inflation attempting to model in more detail the decoherence of the quantum field can be found, for example, in [4].

Here, however, only the simplest model of stochastic inflation will be considered [2]. Thus the dynamics of the coarse grained field is determined by a white noise term. Recently, the equations of stochastic inflation for the coarse grained inflaton have been solved explicitly in a perturbative way [5] (for earlier work in this context, see [6]). This allows to take into account as back reaction the evolution of the inflaton, namely the time variation of the Hubble parameter. In this work the perturbative approach is used to determine the spectrum of the resulting classical fluctuations due to the coarse grained field. Moreover, it is possible to estimate the amount of non-gaussianity, encoded in the parameter f_{NL} [7]-[11], by including terms up to second order in the perturbative expansion.

2 Curvature perturbation

In the slow roll regime of stochastic inflation the evolution of the coarse grained scalar field φ is described by the Langevin equation [2],

$$\dot{\varphi} + \frac{1}{3H} \frac{dV}{d\varphi} = \frac{H^{\frac{3}{2}}}{2\pi} \xi(t), \quad (2.1)$$

where $\xi(t)$ is a white noise process with $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = \delta(t - t')$. Furthermore, the Hubble parameter is given by $H^2(\varphi) = \frac{8\pi}{3M_P^2} V(\varphi)$. Following [5] equation (2.1) can be solved by the perturbative expansion

$$\varphi(t) = \varphi_c(t) + \delta\varphi_1(t) + \delta\varphi_2(t) + \dots, \quad (2.2)$$

where φ_c is the deterministic background value of the coarse grained field. Using the expansion (2.2) in equation (2.1) results in [5]

$$\frac{d\varphi_c}{dt} = -\frac{M_P^2}{4\pi} H'(\varphi_c) \quad (2.3)$$

$$\frac{d\delta\varphi_1}{dt} = -\frac{M_P^2}{4\pi} H''(\varphi_c) \delta\varphi_1 + \frac{H^{\frac{3}{2}}(\varphi_c)}{2\pi} \xi(t) \quad (2.4)$$

$$\frac{d\delta\varphi_2}{dt} = -\frac{M_P^2}{4\pi} H''(\varphi_c) \delta\varphi_2 - \frac{M_P^2}{8\pi} H'''(\varphi_c) (\delta\varphi_1)^2 + \frac{3}{4\pi} H^{\frac{1}{2}}(\varphi_c) H'(\varphi_c) \delta\varphi_1 \xi(t), \quad (2.5)$$

where a prime denotes the derivative with respect to φ_c . The solutions for the first and second order perturbations are given by [5]

$$\delta\varphi_1(t) = \frac{H'[\varphi_c(t)]}{2\pi} \int_0^t d\tau \frac{H^{\frac{3}{2}}[\varphi_c(\tau)]}{H'[\varphi_c(\tau)]} \xi(\tau) \quad (2.6)$$

$$\delta\varphi_2(t) = H'[\varphi_c(t)] \int_0^t d\tau \left[-\frac{M_P^2}{8\pi} \frac{H'''[\varphi_c(\tau)]}{H'[\varphi_c(\tau)]} (\delta\varphi_1)^2 + \frac{3}{4\pi} H^{\frac{1}{2}}[\varphi_c(\tau)] \delta\varphi_1 \xi(\tau) \right]. \quad (2.7)$$

The spectrum induced by the first order fluctuation $\delta\varphi_1$ can be calculated using the following expression for the curvature fluctuation, e.g. [1],

$$\langle \mathcal{R}^2 \rangle = \left(\frac{H(\varphi_c)}{\dot{\varphi}_c} \right)^2 \langle \delta\varphi_1^2 \rangle. \quad (2.8)$$

Besides, since the curvature perturbation is constant outside the horizon it can be assumed that its spectrum is just a function of wave number. Thus

$$\langle \mathcal{R}^2 \rangle = \int_0^k \mathcal{P}_{\mathcal{R}} \frac{dq}{q} = \int_{-\infty}^{\ln k} \mathcal{P}_{\mathcal{R}} d \ln q. \quad (2.9)$$

where k is some maximal scale, which corresponds to a minimal length scale that can be “probed”. This scale is determined by the coarse graining scale. The spectrum is calculated at the coarse graining scale, $k = \sigma aH$, where σ is a parameter. Thus with $d \ln k = H dt$ the spectrum is given by

$$\mathcal{P}_{\mathcal{R}} = \frac{1}{H} \frac{d}{dt} \langle \mathcal{R}^2 \rangle, \quad (2.10)$$

see for example [12]. Using the solution for $\delta\varphi_1$ results in

$$\mathcal{P}_{\mathcal{R}} = \left(\frac{4\pi}{M_P^2} \right)^2 \left(\frac{H}{H'} \right)^2 \frac{H^2}{4\pi^2} - \frac{8}{M_P^4} (H')^2 \int_{\varphi_c}^{\varphi_0} d\psi_c \left(\frac{H}{H'} \right)^3, \quad (2.11)$$

where $\varphi_0 \equiv \varphi_c(0)$. Clearly, the first term is the standard result of the curvature perturbation spectrum due to a quantum field in a de Sitter background. The correction term is due to the evolution of the Hubble parameter in time. As can be shown, assuming H to be constant in time results in the correction term vanishing. In the standard approach the change due to the evolution of the inflaton is neglected. The variation in H is being considered only when deriving the spectral index and derived quantities of that (see for example [1]). Moreover, choosing φ_0 to be the value of the coarse grained field at horizon crossing, it is found that for the choice of the coarse graining scale to be of the order of the horizon there is no correction of the spectral amplitude. This is consistent with the picture that the spectrum in the standard approach is calculated at horizon crossing. In case that the coarse graining scale and the horizon are not the same equation (2.11) implies that the spectrum is reduced with respect to the standard result. Modes with wavelengths between the horizon and the coarse graining scale are not taken into account. Thus the amplitude of the spectrum in the stochastic approach is in general reduced with respect to the amplitude in the standard approach.

In the following the potential is assumed to be of the form

$$V(\varphi_c) = V_0 \left(\frac{\varphi_c}{M_P} \right)^n. \quad (2.12)$$

The mean square fluctuation at first order is found to be

$$\langle \delta\varphi_1(t)^2 \rangle = \frac{H^2}{4\pi^2} \frac{2\pi}{n} \left(\frac{\varphi_c}{M_P} \right)^2 \left[\left(\frac{\varphi_0}{\varphi_c} \right)^4 - 1 \right]. \quad (2.13)$$

In comparison, the amplitude of quantum fluctuations of a massless free scalar field in de Sitter space-time is given by $\delta\phi = H/2\pi$. However, it can be shown that at a time $t = H^{-1}$ the mean

square first order fluctuation is given by $\langle \delta\varphi_1^2(H^{-1}) \rangle \simeq \frac{H^2}{4\pi^2}$. This is consistent with the general picture that the stochastic differential equation (2.1) describes a random walk process with a step size of the order of $H/2\pi$ over a characteristic time H^{-1} [13]. Here, this holds at first order in the expansion of the perturbation.

The power spectrum of the curvature fluctuations $\mathcal{P}_{\mathcal{R}}$ for the potential (2.12) is found to be

$$\mathcal{P}_{\mathcal{R}} = \frac{32\pi}{3n^2} \left(\frac{V_0}{M_P^4} \right) \left(4 - n \left[\left(\frac{\varphi_0}{\varphi_c} \right)^4 - 1 \right] \right) \left(\frac{\varphi_c}{M_P} \right)^{n+2}. \quad (2.14)$$

This can also be written as $\mathcal{P}_{\mathcal{R}} = \frac{2}{3} \left(4 - n \left[\left(\frac{\varphi_0}{\varphi_c} \right)^4 - 1 \right] \right) \frac{1}{\epsilon} \left(\frac{V}{M_P^4} \right)$, where $\epsilon \equiv \frac{M_P^2}{16\pi} \left(\frac{V'}{V} \right)^2$ is one of the slow roll parameters. In this case the spectrum depends explicitly on the value φ_0 of the coarse grained field. Besides, the requirement $\mathcal{P}_{\mathcal{R}} > 0$ imposes an upper bound on the fraction $\left(\frac{\varphi_0}{\varphi_c} \right)$, implying the range $1 \leq \left(\frac{\varphi_0}{\varphi_c} \right) < \left(1 + \frac{4}{n} \right)^{\frac{1}{4}}$. Thus this gives an upper bound on the size of the coarse graining domain. However, in principle, there should not be any limit. The upper cut-off is due to the fact that this is a perturbative approach and here only the first order is considered. Moreover, considering times of the order of the Hubble time leads to a change in the coarse grained scalar field of the order

$$\frac{\varphi_0}{\varphi_c} \simeq 1 + \frac{n}{8\pi} \left(\frac{\varphi_c}{M_P} \right)^{-2}, \quad (2.15)$$

which is much smaller than the maximal bound imposed by the amplitude of the spectrum. The stochastic equation (2.1) is derived for de Sitter space-time. Using the stochastic approach to inflation to find the effect on the spectrum due to the evolution of the Hubble parameter can be interpreted as a perturbation around the de Sitter case. This is done varying the size of the coarse graining domain, that is varying the ratio $\frac{\varphi_0}{\varphi_c}$.

For comparison, the spectrum in the standard approach is given by $\mathcal{P}_{\mathcal{R}} = \frac{8}{3M_P^4} \frac{V}{\epsilon}$ [1]. Expression (2.14) reduces to this for $\left(\frac{\varphi_0}{\varphi_c} \right) = 1$. The ratio of the amplitude of the curvature fluctuations in the stochastic approach over the one in the standard approach is shown in figure 1 (a). For $\left(\frac{\varphi_0}{\varphi_c} \right) > 1$, the amplitude in the stochastic approach is always lower than in the standard approach.

Whereas the spectral index n_s given by

$$n_s = 1 + \frac{d \ln \mathcal{P}_{\mathcal{R}}}{d \ln k} \quad (2.16)$$

is found to be

$$n_s = 1 - \frac{n}{8\pi} \frac{(n+2)(n+4) - n(n-2) \left(\frac{\varphi_0}{\varphi_c} \right)^4}{n+4 - n \left(\frac{\varphi_0}{\varphi_c} \right)^4} \left(\frac{\varphi_c}{M_P} \right)^{-2}, \quad (2.17)$$

the running of the spectral index is given by

$$\frac{dn_s}{d \ln k} = -(n_s - 1)^2 + \frac{n^2}{64\pi^2} \frac{n(n+2)(n+4) - n(n-2)(n-4) \left(\frac{\varphi_0}{\varphi_c} \right)^4}{n+4 - n \left(\frac{\varphi_0}{\varphi_c} \right)^4} \left(\frac{\varphi_c}{M_P} \right)^{-4}. \quad (2.18)$$

The quantities (2.14) to (2.18) are calculated at $\varphi_c = \varphi_W$ which is the field value N_W e-folds before the end of inflation, when the characteristic scale of WMAP is of the order of the size of the coarse graining domain. φ_W is given by

$$\left(\frac{\varphi_W}{M_P}\right) = \sqrt{\frac{n}{4\pi}N_W + \frac{n^2}{16\pi}}. \quad (2.19)$$

The spectral index in the standard approach [1], $n_s = 1 - 6\epsilon + 2\eta$, with $\eta \equiv \frac{M_P^2}{8\pi} \frac{V''}{V}$ the second slow roll parameter, is given by

$$n_{s,stand} = 1 - \frac{n}{8\pi}(n+2) \left(\frac{\varphi_c}{M_P}\right)^{-2}. \quad (2.20)$$

The spectral index calculated in the stochastic approach is shown in figure 1 (b) for a range of values for N_W . The exact value of N_W depends on the history of the universe. Typically N_W is assumed to be the number of e-folds before the end of inflation when the present Hubble scale left the horizon, which should be a good approximation here. In [14] it has been argued that the maximum range is given by $14 < N_W < 75$. Other authors assume N_W to be in the range between 50 and 60, see for example, [15], or between 46 and 60 [16]. To see how values change in the case at hand we have chosen N_W to be between 45 and 65. It is found that, for a given value of N_W , the larger the value φ_0 with respect to φ_W the smaller the spectral index.

Assuming negligible running of the spectral index and negligible contributions from tensor perturbations the spectral index is found to be [17] $n_s = 0.951 \pm 0.016$ from the three year WMAP data only, $n_s = 0.948 \pm 0.015$ from WMAP data and data from the 2dF Galaxy Redshift Survey (2df) and $n_s = 0.948_{-0.015}^{+0.016}$ from WMAP data and data from the Sloan Digital Sky Survey (SDSS). In general the spectral index in the stochastic approach depends on $\frac{\varphi_0}{\varphi_W}$ and N_W . It is always smaller than the one obtained in the standard approach, which only depends on N_W . This is true even at $\frac{\varphi_0}{\varphi_W} = 1$. Furthermore, it is found that only the spectral index for a potential with $n = 2$ is compatible with observations. Moreover, in the case at hand the observational constraints can be satisfied only for $N_W = 55$ and $N_W = 65$, requiring $\frac{\varphi_0}{\varphi_W} < 1.08$ for $N_W = 55$ and $\frac{\varphi_0}{\varphi_W} < 1.12$ for $N_W = 65$.

The running of the spectral index in the standard approach [1], $dn_{s,stand}/d \ln k = 16\epsilon\eta - 24\epsilon^2 - 2\xi_{sl}^2$, where $\xi_{sl}^2 \equiv \frac{M_P^4}{64\pi^2} \frac{V'}{V} \frac{V'''}{V}$ is a third slow roll parameter, is given by

$$\frac{dn_{s,stand}}{d \ln k} = -\frac{n^2(n+2)}{32\pi^2} \left(\frac{\varphi_c}{M_P}\right)^{-4}. \quad (2.21)$$

The running of the spectral index in the stochastic approach is shown in figure 1(c). Thus as in the standard approach, the running of the spectral index calculated in the stochastic approach is negative. For a chosen value of N_W , its modulus is increasing with increasing values of $\frac{\varphi_0}{\varphi_W}$. In figure 2 we have plotted in the case of a quadratic potential the running of the spectral index $dn_s/d \ln k$ versus the spectral index n_s in the stochastic as well as the standard approach. As can be seen in this figure the corresponding values for the running of the spectral index are much below observational bounds, assuming negligible contributions from tensor perturbations, $\frac{dn_s}{d \ln k} = -0.055_{-0.031}^{+0.030}$ from WMAP three year data only, $\frac{dn_s}{d \ln k} = -0.048 \pm 0.027$ from WMAP and 2df data and $\frac{dn_s}{d \ln k} = -0.060 \pm 0.028$ from WMAP and SDSS data [17]. Thus the assumption of negligible running of the spectral index is well justified. Similar results are found for the contribution from

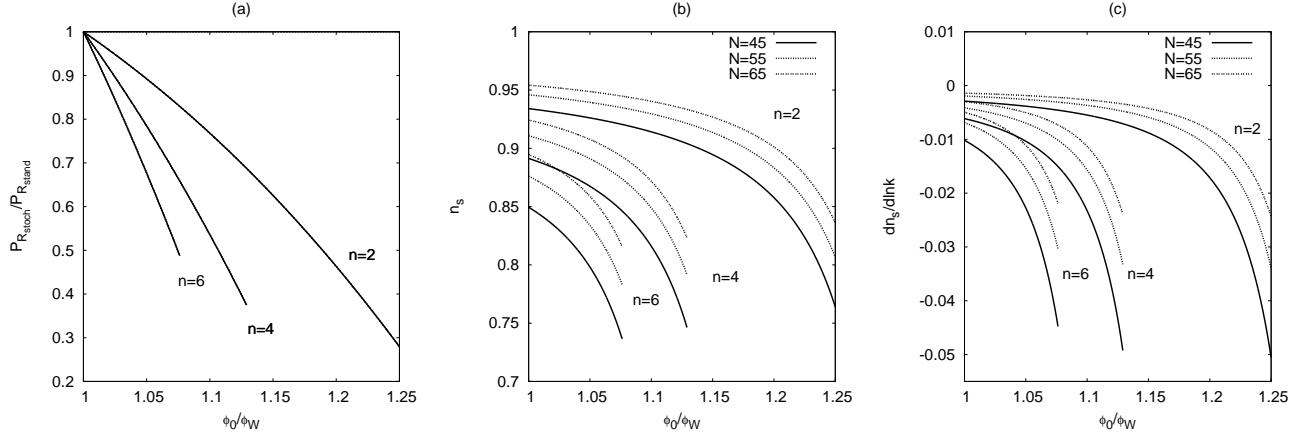


Figure 1: (a) The ratio of the amplitude of the curvature perturbation due to the stochastic over the amplitude due to the standard approach $\frac{P_{\mathcal{R}_{stoch}}}{P_{\mathcal{R}_{stand}}}$. (b) The spectral index in the stochastic approach. (c) The running of the spectral index in the stochastic approach. Graphs are shown for $N_W = 45, 55, 65$. Furthermore, n denotes the power of the potential.

tensor perturbations (see next section) as can be seen from figure 3 (b), which shows the tensor-to-scalar amplitude ratio versus the spectral index in the case of a quadratic potential in the standard and in the stochastic approach. Thus in order to satisfy observational bounds $\frac{\varphi_0}{\varphi_W}$ has to be close to 1. Therefore deviations from de Sitter space-time are small.

3 Gravitational wave spectrum

In this section the tensor-to-scalar amplitude ratio r is calculated which is another observational parameter. The gravitational wave amplitudes h_{ij} are effectively determined by two massless scalar fields ψ_+ , ψ_\times , respectively [1]. In particular, $h_{ij} = h_+ e_{ij}^+ + h_\times e_{ij}^\times$, where e^+ and e^\times are polarization tensors and $h_{+, \times} = \left(\frac{M_P^2}{16\pi}\right)^{-\frac{1}{2}} \psi_{+, \times}$. For simplicity, the index of $\psi_{+, \times}$ is omitted and the scalar fields are simply denoted by ψ . Assuming that ψ is in the slow roll regime, at first order the following equation holds

$$\dot{\psi} = \frac{H^{\frac{3}{2}}(\varphi_c)}{2\pi} \xi(t), \quad (3.22)$$

where the noise $\xi(t)$ is defined as before. This is solved by

$$\psi(t) = \frac{1}{2\pi} \int_0^t d\tau H^{\frac{3}{2}}(\tau) \xi(\tau), \quad (3.23)$$

implying

$$\langle \psi^2(t) \rangle = \frac{1}{4\pi^2} \int_0^t d\tau H^3(\tau). \quad (3.24)$$

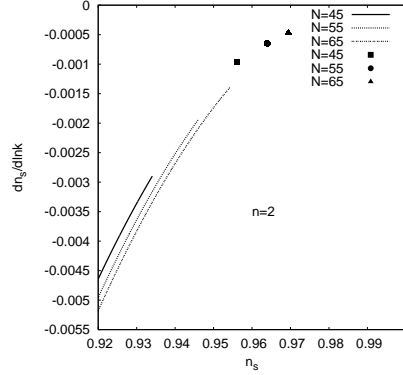


Figure 2: The graph shows the running of the spectral index against the spectral index in the stochastic approach (lines) and the standard approach (points) for a quadratic potential for $N_W = 45, 55, 65$.

Thus the spectrum of gravitational waves, $\mathcal{P}_{grav} = \frac{1}{H} \frac{d\langle h_{ij} h^{ij} \rangle}{dt}$, is found to be

$$\mathcal{P}_{grav} = \frac{16}{\pi} \left(\frac{H}{M_P} \right)^2, \quad (3.25)$$

which equals the spectrum in the standard approach (cf. [1], e.g.). Hence the tensor-to-scalar amplitude ratio $r \equiv \frac{\mathcal{P}_{grav}}{\mathcal{P}_{\mathcal{R}}}$ is found to be

$$r_{stoch} = \frac{16\epsilon}{1 - \frac{n}{4} \left[\left(\frac{\varphi_0}{\varphi_c} \right)^4 - 1 \right]}. \quad (3.26)$$

Therefore, for $\left(\frac{\varphi_0}{\varphi_c} \right) = 1$ the tensor-to-scalar amplitude ratio in the standard approach, $r_{stand} = 16\epsilon$, (see for example [10]) is recovered. For all other values, $r_{stoch} > r_{stand}$. This is due to the fact that in the stochastic approach the curvature spectrum amplitude is lower than in the standard approach.

The tensor-to-scalar amplitude ratio calculated in the stochastic approach is shown in figure 3 (a) for different values of N_W . Assuming negligible running of the spectral index observational bounds on r are found to be: $r < 0.65$ from the three year WMAP data only, $r < 0.38$ from WMAP and 2df data and $r < 0.30$ from WMAP and SDSS data ([17] (for the last value see also [16])). The constraint from the WMAP three year data only admits all three types of potentials considered in figure 3 (a), up to a certain value of $\frac{\varphi_0}{\varphi_W}$. The constraint from WMAP and 2df data marginally allows a quartic potential with $N_W = 45$, which is already excluded by the WMAP plus SDSS data, independent of $\frac{\varphi_0}{\varphi_W}$. This stronger observational constraint can be satisfied up to a certain value of $\frac{\varphi_0}{\varphi_W}$ for potentials with $n = 4$ and $N_W > 45$ and for quadratic potentials. In figure 3 (b) the tensor-to-scalar amplitude ratio versus the spectral index is shown for a quadratic potential calculated in the stochastic as well as the standard approach. Values for r allowed by the observational constraints on the spectral index are of the same order in the stochastic and in the standard approach.

4 Non-gaussianity

By construction, the first order perturbation $\delta\varphi_1$ is gaussian with zero mean, $\langle\delta\varphi_1\rangle = 0$. Thus the curvature perturbation \mathcal{R} is gaussian at first order. Non-gaussianity enters in \mathcal{R} at second order in the expansion of $\delta\varphi$ (cf. equation (2.2)). The nonlinearity can be estimated using the parameter f_{NL} defined by [7]

$$\mathcal{R} = \mathcal{R}_g - \frac{3}{5}f_{NL}(\mathcal{R}_g^2 - \langle\mathcal{R}_g^2\rangle), \quad (4.27)$$

where \mathcal{R}_g is a gaussian distribution with $\langle\mathcal{R}_g\rangle = 0$. Thus by construction $\langle\mathcal{R}\rangle = 0$. In order to determine the non-gaussian contribution due to the second order perturbation, it is convenient to consider the curvature perturbation $\mathcal{R} = -\frac{H}{\dot{\varphi}_c}\delta\varphi$, where $\delta\varphi = \delta\varphi_1 + \delta\varphi_2 - \langle\delta\varphi_2\rangle$. Hence by construction $\langle\mathcal{R}\rangle = 0$. Furthermore, the gaussian contribution is given by

$$\mathcal{R}_g \equiv -\frac{H}{\dot{\varphi}_c}\delta\varphi_1. \quad (4.28)$$

Thus f_{NL}^2 is found to be

$$f_{NL}^2 = \frac{25}{9} \left(\frac{M_P^2}{4\pi} \right)^2 \left(\frac{H'}{H} \right)^2 \frac{\langle\delta\varphi_2^2\rangle - \langle\delta\varphi_2\rangle^2}{\langle\delta\varphi_1^4\rangle - \langle\delta\varphi_1^2\rangle^2}. \quad (4.29)$$

Taking into account only the dominant contribution in $\delta\varphi_2$, coming from the second term in equation (2.7), yields to

$$f_{NL}^2 \simeq \frac{25}{128\pi^2} \left(\frac{H}{M_P} \right)^{-2} \frac{\int_{\varphi_c}^{\varphi_0} d\psi H H' \int_{\psi}^{\varphi_0} d\chi \left(\frac{H}{H'} \right)^3 + \frac{M_P}{4\pi} \int_{\varphi_c}^{\varphi_0} d\psi \frac{H^4}{H'}}{M_P^{-2} \left(\int_{\varphi_c}^{\varphi_0} d\psi \left(\frac{H}{H'} \right)^3 \right)^2}. \quad (4.30)$$

This expression is exact for potentials with index $n = 2$ and $n = 4$, since in these cases $H''' \equiv 0$. For a potential of the form $V = V_0 \left(\frac{\varphi_c}{M_P} \right)^n$ equation (4.30) leads to, at $\varphi_c = \varphi_W$,

$$\begin{aligned} f_{NL}^2 \simeq & \frac{25}{128\pi^2} \left[\frac{n^4}{4} \left[\frac{4}{n(n+4)} \left(\frac{\varphi_0}{\varphi_W} \right)^{n+4} - \frac{1}{n} \left(\frac{\varphi_0}{\varphi_W} \right)^4 + \frac{1}{n+4} \right] \left(\frac{\varphi_W}{M_P} \right)^{-4} \right. \\ & \left. + \frac{n^5}{4\pi(3n+4)} \left(\frac{8\pi}{3} \right)^{\frac{1}{2}} \left(\frac{V_0}{M_P^4} \right)^{\frac{1}{2}} \left(\frac{\varphi_W}{M_P} \right)^{\frac{n}{2}-6} \left[\left(\frac{\varphi_0}{\varphi_W} \right)^{\frac{3n}{2}+2} - 1 \right] \left[\left(\frac{\varphi_0}{\varphi_W} \right)^4 - 1 \right]^{-2} \right] \end{aligned} \quad (4.31)$$

The parameter $|f_{NL}|$ is shown in figure 3 (c) for several values of N_W and the potential index n . An estimate for $\left(\frac{V_0}{M_P^4} \right)$ is found using the expression for the power spectrum amplitude $\mathcal{P}_{\mathcal{R}}$ (cf. equation (2.14)) together with $\mathcal{P}_{\mathcal{R}} = 2.4 \times 10^{-9}$ [17]. As can be seen from figure 3 (c), f_{NL} satisfies the bound from observations, $-54 < f_{NL} < 114$ [18].

For $n = 4$ the parameter $|f_{NL}|$ only depends very weakly on $\frac{\varphi_0}{\varphi_W}$. For potentials with an index $n > 4$ the parameter $|f_{NL}|$ is increasing with increasing values of $\frac{\varphi_0}{\varphi_W}$. For $n = 2$ $|f_{NL}|$ is a decreasing function for increasing values of $\frac{\varphi_0}{\varphi_W}$. The non-gaussian contribution requires that $\frac{\varphi_0}{\varphi_W} > 1$.

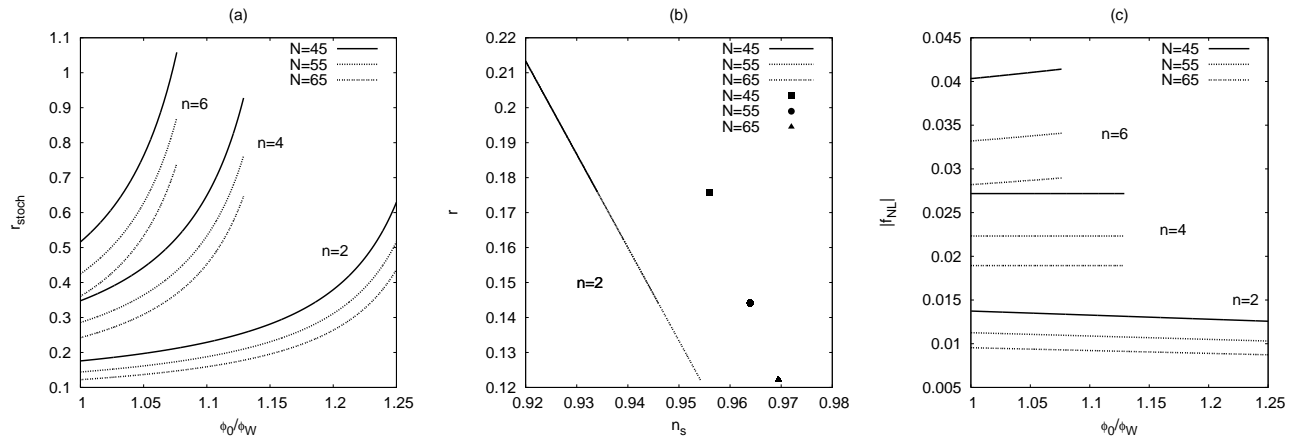


Figure 3: (a) The tensor-to-scalar amplitude ratio r_{stoch} in the stochastic approach. (b) The tensor-to-scalar amplitude ratio against the spectral index in the stochastic (lines) and in the standard approach (points) in the case of a quadratic potential. (c) The non-gaussianity parameter $|f_{NL}|$ in the stochastic approach. Curves are shown for $N_W = 45, 55, 65$. Furthermore, n denotes the power of the potential.

These results can be compared with values for f_{NL} found in the standard approach. Perturbation analysis up to second order in the standard approach to slow roll inflation leads to the estimate that f_{NL} is of the order of the slow roll parameters [8]-[11]. In [8] f_{NL} is found to be $f_{NL} \sim \frac{5}{12}(n_s + f(k)n_t)$, where n_t is the spectral index of the gravitational wave spectrum and $0 \leq f \leq \frac{5}{6}$ characterizes the configuration of the momenta which enters into the calculation of the three-point correlation function of the curvature perturbation. Choosing $f_{NL} = \frac{5}{6}$ (equilateral triangle) results for the range of parameters under consideration here ($N_W = 45, 55, 65$ and $n = 2, 4, 6$) in a non-linearity parameter $f_{NL} \sim \mathcal{O}(10^{-2})$. Thus it is of the same order of magnitude as f_{NL} calculated in the stochastic approach (cf. figure 3 (c)).

5 Conclusions

Recently, the equations of stochastic inflation have been solved in a perturbative way [5]. This allows to take into account the effect on the perturbations due to the evolving Hubble parameter. Here, the perturbative approach to stochastic inflation has been used to determine the spectrum of the curvature perturbation due to the coarse grained inflaton as well as the gravitational wave spectrum.

The spectrum of curvature fluctuations has been calculated using the solution for the perturbation of the coarse grained field up to first order in the noise. The resulting expression for the spectrum contains an additional parameter in the form of a value φ_0 of the coarse grained scalar field φ . This has been interpreted as the value of the deterministic coarse grained inflaton φ_c at the horizon. This interpretation is motivated by the fact that the resulting perturbation spectrum reduces to the standard expression in the case $\varphi_c = \varphi_0$. In the standard approach the curvature perturbation spectrum is found using the spectrum of vacuum fluctuations in a de Sitter background calculated at the horizon. The coarse graining of the scalar field in the stochastic approach to inflation introduces a coarse graining scale. This is at least of the order of the horizon. The spectrum is calculated

at the coarse graining scale. In general the amplitude of the spectrum of curvature fluctuations calculated using the stochastic approach is reduced, in comparison with the standard approach, for $\frac{\varphi_0}{\varphi_c} > 1$. This results from the horizon and the coarse graining scale not being the same in this case. Modes with wavelengths between the horizon and the coarse graining scale do not contribute to the spectrum.

The expression for the spectral index depends in general on the parameter φ_0 . Even at $\varphi_c = \varphi_0$ the spectral index is smaller than the one obtained in the standard approach. Moreover, only for polynomial potentials with a potential index $n = 2$ observational bounds can be satisfied. The running of the spectral index calculated in the stochastic approach is negative as it is the case in the standard approach. Apart from its dependence on the index of the potential and the number of e-folds before the end of inflation, at which the coarse grained field is evaluated, its value also depends on $\frac{\varphi_0}{\varphi_c}$.

The tensor-to-scalar amplitude ratio is in general larger in the stochastic than in the standard approach for $\frac{\varphi_0}{\varphi_c} > 1$. This is due to the fact that whereas the gravitational wave amplitude is found to be the same as in the standard approach the scalar perturbation amplitude is lower in the stochastic approach.

Finally, the non-gaussian contribution due to the second order perturbation in the stochastic approach to inflation has been calculated. By construction perturbations at first order in the noise are gaussian since the noise is gaussian. Non-gaussianity enters at second order in the noise. The resulting expression for the parameter $|f_{NL}|$ requires $\frac{\varphi_0}{\varphi_c} > 1$. For potentials with a potential index $n = 2$ it is a decreasing function with increasing $\frac{\varphi_0}{\varphi_c}$, for $n = 4$ it is only very weakly depending on the value of $\frac{\varphi_0}{\varphi_c}$ and for $n > 4$ it is an increasing function with $\frac{\varphi_0}{\varphi_c}$. The resulting values for $|f_{NL}|$ are compatible with bounds from observations.

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